ABSTRACT
Many companies show an interest to collect used products to recover residual value because of environmental regulations and consumer pressures. Reverse logistics can lead to better relationships among supply chain partners, more profits through reduced costs, and greater efficiencies and higher remanufacturing rates for used products. Price discount is a significant approach for developing a closed-loop supply chain coordination policy. The joint decision policy characterized by selling prices, production quantity and replenishment schedules is coordinated through price discount. This study analyzes a closed-loop supply chain inventory system with inventory stages in recycles. In addition to traditional forward material flows, the used products are returned to a recovery facility where they can be stored, remanufactured and shipped to retailers for sale. This study investigates a price discount policy for a multi-echelon inventory system with remanufacturing and price sensitive demand for deteriorating items. We derive the optimal price discount policy by considering the integration of the manufacturer and the retailer, and develop a mathematical closed-loop supply chain inventory model with remanufacturing and price dependent demand for deteriorating items. An example of a closed-loop multi-echelon inventory model is used to illustrate the system with different negotiation factors and profit-sharing ratios. Thus, obtaining the greatest benefits for the whole system should be in a cooperative environment. Price discount policy must be incorporated into the supply chain inventory management system.

KEY WORDS
Closed-loop Supply Chain, Multi-echelon Inventory, Reversed Logistics, Deteriorating Items, Price Discount

1. Introduction
In order to reduce cost and conform to environmental regulations, many companies not only collect used products and but also design their products to allow for reuse and remanufacture. Product remanufacturing such as transforming used items into usable products through refurbishment, repair and upgrading plays an important role in the production, distribution and retailing system. One of the principal objectives of a supply chain management is to develop a positive design mechanism to coordinate the manufacturer and the retailer. The cooperative decision policy characterized by the selling price, the ordering cycle time and the replenishment schedule is coordinated through price discounting. Many researchers have found ways of managing forward-oriented supply chains and had fascinating insights in handling single-stage inventory systems.

Schrady [1], the earliest author, proposed a deterministic model with infinite production rates with manufacturing and remanufacturing. The optimal ordering lot sizes for the manufacturing and remanufacturing were similar to the classical EOQ formula. Several authors extended the model of Schrady. Nahmias and Rivera [2] generalized this model for finite remanufacturing rates; Mabini et al. [3] considered a stock out service level constraints and the multi-product model. With a different control policy, Richter [4] suggested a model which was similar to Schrady’s model. Koh, et al. [5] analyzed situations where manufacturing/remanufacturing rates were finite. Several other studies separately addressed collection issues. Savaskan et al. [6] determined the optimal collection channel configuration of a monopolist manufacturer. Bautista and Pereira [7] emphasized locating these collection area problems, and established the relationship between the set covering problem and the MAX-SAT problem.
Because the delivery number and lot size would influence an overall integrated cost, Monahan [8] investigated the integration approach by minimizing total costs for the vendor and discounting the price offered by the buyer to reach a compromise between the two parties. Lu [9] developed a one-vendor multi-buyer integrated model to minimize vendor price and maximum the acceptable cost to the buyer. Goyal [10] designed a policy in which buyer delivery lot size comprised the entire inventory available to the vendor. Hill [11] offered a more general type of policy for a single-vendor, single buyer production-inventory model based on successive shipments to the buyer within a single production batch size increasing by a fixed factor. Yang and Wee [12] developed an arborescent model that considered the integration of the producer, distributor and retailer. Chung and Wee [13] proposed an optimal replenishment and pricing policy by maximizing the supplier, manufacturer and retailer’s shared profit in a price-sensitive demand.

Deterioration is a significant consideration in the inventory model. Ghare and Schrader [14] were the first authors to study inventory deterioration with constant demand. Other authors, Wee [15], Yang and Wee [16] and Wee, Yang and Shih [17], established their models based on multifarious assumptions. Yang and Wee [18] and Yang [19] derived a model for integrating the vendor and the buyer to optimize the delivery number and lot size for deteriorating items,

This study analyses a closed-loop supply chain inventory system with remanufacturing for deteriorating items. In addition to the traditional forward material flows, the used products are returned to recovery facilities where they will be remanufactured and shipped to retailers for sale. The situation is illustrated in Figure 1. The objective of this study is to derive an optimal replenishment and pricing policy by maximizing the retailer and manufacturer’s joint profit in a win-win cooperation. We derive the optimal policy by considering the integration of the manufacturer and the retailer, and develop a mathematical model that simultaneously determines the discount price, the number of deliveries and the replenishment interval with exponentially decreasing price dependent demand. An example of a closed-loop multi-echelon inventory model is used to illustrate the system with different negotiation factors and profit-sharing ratios. That is the extra profit accrued mainly to the down-stream site should be shared. One can show that the integrated approach results in an impressive extra profit for the whole system.

2. Assumptions and Notation

The mathematical models in this paper are developed on the basis of the following assumptions:

(1). An infinite planning horizon.
(2). Deterioration is considered.
(3). Shortage of stock is not allowed.
(4). The manufacturing and remanufacturing rate of the manufacturer are finite and the lead-time is constant.
(5). A single item is considered in a closed-loop two-echelon inventory system.

(6). The annual demand rate and the annual return rate are exponentially decreasing price dependent.
(7). There are no constraints on space, capacity or capital.
(8). The number of deliveries is an integer.
(9). The setup cost per run, and the annual holding cost fraction are known and constant.
(10). The manufacturer serves the retailer’s demand as far as possible from remanufactured products.
(11). The remanufactured products are the same as manufactured new products.
(12). Single manufacturer and single retailer are considered.

The following denotes the parameters and variables for the retailer:

\[ \theta \] the deterioration rate, \( 0 < \theta < 1 \);
\[ A_r \] the ordering cost for the retailer;
\[ F_r \] the inventory holding cost percentage per year per dollar for the retailer;
\[ p_c \] the retailer selling price to the consumer (decision variable);
\[ P_c \] the retailer selling price to the consumer before pricing discounts;
\[ p_r \] the retailer purchase unit price from the manufacturer (decision variable);
\[ P_r \] the retailer purchase unit price from the manufacturer before pricing discounts;
\[ d \] the annual price-sensitive demand rate (\( d = a e^{-\theta p_r} \));
\[ D \] the annual price-sensitive demand rate before pricing discounts (\( D = a e^{-\theta p_r} \));
\[ T_r \] the retailer ordering cycle time (decision variable);
\[ TP_r \] the total profit per unit time for the retailer;
\[ TC_r \] the total relevant cost per unit time for the retailer.

The following denotes the parameters and variables for the manufacturer:

\[ A_M \] the setup cost per run for manufacturing;
\[ A_k \] the setup cost per run for remanufacturing;
\[ F_M \] the product inventory holding cost percentage per year per dollar for the manufacturer;
\[ P_M \] the manufacturer purchase unit price from the supplier;
\[ p_R \] the manufacturer purchase unit price from the consumer (\( p_R = R p_c \));
\[ T_{R1} \] the manufacturer reproduction period in each cycle;
\[ T_{R2} \] the manufacturer non-reproduction period in each cycle;
\[ T_{M1} \] the manufacturer production period in each cycle;
\[ T_{M2} \] the manufacturer non-production period in each cycle;
\[ m \] the number of deliveries per production cycle time from the manufacturer to the retailer (decision variable);
\[ n \] the number of deliveries per reproduction cycle time from the manufacturer to the retailer (decision variable);
\[ I \] the number of deliveries per reproduction/production cycle time from the manufacturer to the retailer, \( I = m + n \), where \( I \) is a positive integer;
\[ M \] the fixed cost to process buyer’s order of any size for manufacturer;
\[ P \] the annual production rate of the manufacturer (\( P > D \));
\[ R \] the annual reproduction rate of the manufacturer (\( R > D \));
\[ c \] annual return rate (\( c = C_d \));
\[ TC_m \] the total relevant cost per unit time for the manufacturer;
\[ TP_m \] the total profit per unit time for the manufacturer.

The following denotes the objective function for the optimization problem:

\[ JP(p_r, p_c, n, m, T_r) \] the joint profit per unit time for the whole system.

### 3. Model Development

For a continuous review policy, Figure 2 represents the manufacturer-retailer inventory system. The narrow solid line denotes the stock of the manufacturer, the narrow dash line represents the stock of the retailer, and the wide line shows the total stock of the manufacturer and the retailer. As shown in Figure 2, the retailer’s product inventory differential equation can be formulated as

\[
\frac{dI_r(t)}{dt} + \theta I_r(t) = -d, \quad 0 \leq t \leq T_r
\]

From the boundary condition \( I(T_r) = 0 \), the above differential equations can be solved as follows:

\[
I_r(t) = \frac{d}{\theta} \left( e^{\theta T_r t} - 1 \right), \quad 0 \leq t \leq T_r
\]

and that the retailer’s order quantity \( Q_r = I_r(0) \) one has

\[
I_r(0) = \frac{d}{\theta} \left( e^{\theta T_r} - 1 \right)
\]

The average inventory level of the retailer \( (AIL_r) \) is

\[
AIL_r = \frac{1}{T_r} \int_0^{T_r} I_r(t) dt = \frac{d}{T_r \theta^2} \left( e^{\theta T_r} - 1 \right) - \frac{d}{\theta}
\]

The retailer total relevant annual cost function \( TC_r(T_r) \) can be derived as:

\[
TC_r(p_r, p_c, T_r) = SC_r + HC_r + PC_r
\]

\[
= \frac{A_r}{T_r} + \frac{p_c F_r}{T_r \theta^2} \left( d e^{\theta T_r} - d - dT_r \theta \right) + \frac{p_r d}{T_r \theta} \left( e^{\theta T_r} - 1 \right)
\]

(5)

The first term is the ordering cost, the second term is the holding cost and the third term is the purchasing cost (including the deteriorated cost). The retailer’s total profit is the sales revenue minus the retailer’s total cost. It is shown as follows:
When coordination is considered, Figure 2 shows that

\begin{equation}
\theta = \frac{d}{\theta} \left( e^{\theta t_0} - 1 \right)
\end{equation}

(7)

The amount of stock required by the retailer’s during the reproduction of first shipment, \( Q_r \). The reproduction stock increases at the reproduction rate of \( R \) during the time interval \( t_0 \) and its quantity equals the amount of the first shipment. The equation is shown as follows:

\begin{equation}
\frac{R}{\theta} \left( 1 - e^{-\theta t_0} \right) = Q_r = \frac{d}{\theta} \left( e^{\theta t_0} - 1 \right)
\end{equation}

(8)

Solve (8) respect to \( t_0 \), and the value of \( t_0 \) can be derived as:

\begin{equation}
t_0 = -\frac{1}{\theta} \ln \left[ 1 - \frac{d}{R} \left( e^{\theta t_0} - 1 \right) \right]
\end{equation}

(9)

Substituting \( t_0 \) from (9) into (7) the value of \( g \) can be derived as:

\begin{equation}
g = \frac{d^2}{\theta} \left( \frac{e^{\theta t_0} - 1}{R + d - d e^{\theta t_0}} \right)
\end{equation}

(10)

Figure 2: The Integrated Manufacturer and Retailer Multi-Delivery Inventory Model with Remanufacturing

The total stock increases at the rate of \( (R - d) \) during the \( T_{R1} \), and decreases at the rate of \( d \) during the \( T_{R2} = nT_r \), the maximum inventory level is

\begin{equation}
g + \frac{(R-d)}{\theta} \left( 1 - e^{-\theta t_0} \right) = g + \frac{d}{\theta} \left( e^{\theta (nT_r - t_0)} - 1 \right)
\end{equation}

(11)

Solve (11) respect to \( T_{R1} \), and the value of \( T_{R1} \) can be derived as:

\begin{equation}
T_{R1} = \frac{1}{\theta} \ln \left[ \frac{R - d \left( 1 - e^{\theta t_0} \right)}{R} \right]
\end{equation}

(12)

The remanufacture of the batch quantity is

\begin{equation}
Q_R = R T_{R1} = \frac{R}{\theta} \ln \left[ \frac{R - d \left( 1 - e^{\theta t_0} \right)}{R} \right]
\end{equation}

(13)

Similarly, the manufacture of the batch quantity is

\begin{equation}
Q_M = \frac{P}{\theta} \ln \left[ \frac{P - d \left( 1 - e^{\theta t_0} \right)}{P} \right]
\end{equation}

(14)

The average total stock in remanufacturing period of manufacturer-retailer (AILR) inventory system is
\[ AIL_{Mr} = \left\{ g + \frac{RT_{T_r} - d(T_{T_r} + T_{T_r})}{nT, \theta} \right\} \frac{n}{n + m} \]

Since \( nT_r = T_{T_r} + T_{T_r} \)

\[ = \frac{n}{n + m} \left\{ \frac{d^2}{\theta} \left[ e^{\theta r} - 1 \right] \right\} \frac{R}{R + d(1 - e^{\theta r})} \]
\[
\quad + \frac{R}{\theta^2 nT_r} \ln \left[ \frac{R - d(1 - e^{\theta r})}{R} \right] - \frac{d}{\theta} \]

Similarly, the average total stock in the manufacturing period of manufacturer-retailer inventory system (AIL_{Mr}) is

\[ AIL_{Mr} = \frac{m}{n + m} \left\{ \frac{d^2}{\theta} \left[ e^{\theta r} - 1 \right] \right\} \frac{R}{P + d(1 - e^{\theta r})} \]
\[
\quad + \frac{P}{\theta^2 nT_r} \ln \left[ \frac{P - d(1 - e^{\theta r})}{P} \right] - \frac{d}{\theta} \]

The average total stock of the manufacturer-retailer inventory system (AIL_{mr}) is

\[ AIL_{mr} = AIL_{Mr} + AIL_{Mr} \]

(17)

Consequently, average manufacturer inventory level can be determined by subtracting the average retailer inventory level from the average total inventory of the manufacturer-retailer inventory system. The average manufacturer inventory level (AIL_{mr}) is given by

\[ AIL_{mr} = AIL_{mr} - AIL_{r} \]

(18)

In addition, the total relevant cost for the manufacturer is:

\[ TC_m(p_r, n, m, T_r) = A_d + \frac{(n+m)M}{(n+m)T_r} + F_d \left( \frac{np_r + mP_m}{n + m} \right) \]
\[ \times AIL_{Mr} + \left( \frac{P_r Q_r + P_m Q_m}{(n+m)T_r} \right) \]
\[ = A_d + \frac{(n+m)M}{(n+m)T_r} + F_d \left( \frac{np_r + mP_m}{n + m} \right) \]
\[ \times \left( \frac{md^2}{\theta} \left[ e^{\theta r} - 1 \right] \right) \frac{R}{R + d(1 - e^{\theta r})} + \frac{R}{\theta^2 T_r} \ln \left[ \frac{R - d(1 - e^{\theta r})}{R} \right]\]
\[ + \frac{md^2}{\theta} \left[ e^{\theta r} - 1 \right] \frac{R}{P + d(1 - e^{\theta r})} + \frac{P}{\theta^2 T_r} \ln \left[ \frac{P - d(1 - e^{\theta r})}{P} \right] \]
\[ \quad - \frac{(n+m)d}{T_r, \theta^2} \left[ e^{\theta r} - 1 \right] \frac{R}{R} + \frac{np_r R}{\theta(n+m)T_r} \ln \left[ \frac{R - d + de^{\theta r}}{R} \right] \]
\[ + \frac{P_m P}{\theta(n+m)T_r} \ln \left( \frac{P - d + de^{\theta r}}{P} \right) \]\n
(19)

The first term is the setup cost, the second term is the holding cost and the third term is purchasing cost (including the deteriorated cost).

The manufacturer’s total profit is the sales revenue minus the manufacturer’s total reproduction and production relevant inventory cost. It is shown as follows:

\[ TP_m(p_r, n, m, T_r) = \frac{P_r d}{T_r, \theta} \left( e^{\theta r} - 1 \right) - TC_m \]

(20)

The joint annual profit is the sum of the retailer’s and manufacturer’s total annual profit.

\[ JP(p_r, n, m, T_r) = TP_r + TP_m \]

(21)

The manufacturer’s reproduction and production period are set at \( nT_r \) and \( mT_r \), and the reproduction quantity is equal to the return quantity during the period of \((n+m)T_r = IT_r\) where \( I \) is a positive integer; one has

\[ Q_r = \frac{R}{\theta} \ln \left[ \frac{R - d(1 - e^{\theta r})}{R} \right] = \frac{c}{\theta} \left( 1 - e^{-\theta(n+m)\theta r} \right) \]

(22)

By solving the equation (22), \( c = Cd \) and \( I = n + m \), one has

\[ n = \frac{1}{\theta T_r} \ln \left( \frac{1}{d} \left( \frac{Cd}{\theta} \left[ 1 - e^{-\theta r} \right] R - R + d \right) \right) \]

(23)

and

\[ m = I - n \]

(24)

respectively.
The annual demand decreases exponentially with price such as \( d = ae^{-bp} \). When there is no pricing discount \((p_r = p, p_c = P, D = ae^{-bp_c})\), the buyer and manufacturer’s annual profit are

\[
TP_r(T_r) = PD \left( \frac{A_r}{T_r} + \frac{PF_c}{T_r \theta} (D - DT_r \theta + \frac{PD}{T_r \theta} e^{\theta r} - 1) \right)
\]

and

\[
TP_m(n, m, T_r) = \frac{PD}{T_r \theta} (e^{\theta r} - 1)
\]

respectively.

When there is no pricing discount and decentralized decision-making with sequencing optimization from retailer to manufacturer, the retailer’s optimal ordering decision-making with sequencing optimization from manufacturer to retailer, the retailer’s optimal ordering decision-making is

\[
\begin{aligned}
\max_{T_r} & \quad EP_r(T_r) = \frac{PD}{T_r \theta} \left[ \frac{A_r + (n + m)M}{(n + m) \theta r} - F_r \left( \frac{nP_r + mP_m}{(n + m)^2} \right) \\
& \quad \times \left[ \frac{e^{\theta r} - 1}{R} + \frac{nD}{\theta} \ln \left( \frac{R - D(1 - e^{\theta r})}{R} \right) \right] + \frac{mD}{\theta} \ln \left( \frac{P - D(1 - e^{\theta r})}{P} \right) \right] - \frac{(n + m)D}{T_r \theta} (e^{\theta r} - 1) - \frac{P_m R}{\theta (n + m) \theta r} \ln \left( \frac{R - D + De^{\theta r}}{R} \right) \\
& \quad + \frac{P_m P}{\theta (n + m) \theta r} \ln \left( \frac{P - D + De^{\theta r}}{P} \right)
\end{aligned}
\]

The manufacturer's reproduction/production period is \((n+m)T_r = IT_r\) where \(I\) is a positive integer. The optimal solution \(T_r^*\) can be found that satisfies

\[
TP_m\left( I^+ - 1, T_r^* \right) \leq TP_m\left( I^+, T_r^* \right) \geq TP_m\left( I^+ + 1, T_r^* \right)
\]

With known \(I^*\), the optimal value \(n^*\) and \(m^*\) can be obtained by (23) and (24); and the manufacturer’s total profit with individual decision-making is

\[
TP_m(n^*, m^*, T_r^*)
\]

The maximal joint annual profit with no pricing discount is obtained.

\[
JP\left( n^*, m^*, T_r^* \right) = TP_r\left( T_r^* \right) + TP_m\left( n^*, m^*, T_r^* \right)
\]

The retailer’s annual extra profit \(EP_r\) is the retailer’s annual profit with price discount minus the corresponding optimal annual profit without price discount.

\[
EP_r = TP_r(p_r, \ldots, T_r) - TP_r(P_r, P_c, T_r^*)
\]

and the manufacturer’s annual extra profit \(EP_m\) is

\[
EP_m = TP_m(p_r, \ldots, n, m, T_r) - TP_m(P_r, P_c, n^*, m^*, T_r^*)
\]

We relate \(EP_m\) to \(EP_r\) values as

\[
EP_m = a EP_r, \quad a \geq 0
\]

\(a\) represents an instrument of negotiation. When \(a = 0\), it means all extra profit sharing is accrued to the retailer; when \(a = 1\), it implies that the extra profit sharing is equally distributed. A large \(a\) means that profit is accrued mainly to the manufacturer. The optimisation problem is stated as

\[
\begin{aligned}
\max_{n, m, T_r} & \quad JP(p_r, \ldots, n, m, T_r) \\
\text{s. t.} & \quad EP_m = a EP_r, \quad a \geq 0 \\
& \quad EP_r \geq 0 \\
& \quad n = \frac{1}{\theta T_r} \ln \left( \frac{1 - e^{\frac{D T_r}{\theta r} \left( 1 - e^{\theta r} \right)}}{R - R + d} \right) \\
& \quad m = I - n \\
& \quad I = 1, 2, 3, \ldots \quad p_r, p_c, T_r \geq 0
\end{aligned}
\]
4. Solution Procedure

The optimization problem is to determine the value of $p^*$, $p^*$, $n$, $m$ and $T_r$ that maximizes $JP(p^*, p^*, n, m, T_r)$. Since the problem is a constrained nonlinear mixed programming problem, one can derive the value of $p^*$, $p^*$, $n$, $m$, and $T_r$ by the following procedure:

(a) When there is before pricing discount ($p_r = P_r$, $p_c = P_c$ and $D = ae^{-bt}$), (27) and the values of $\bar{I}$ satisfy (30).

The retailer’s optimal ordering cycle time $T^*_r$, the optimal value $n^*$ and $m^*$ before pricing discount can be derived.

(b) The number of $I$ will be in the vicinity of the number of $\bar{I}$ derived a set of the Karush-Kuhn-Tucker conditions

\[
\frac{\partial J_P}{\partial p_r} + \frac{\partial}{\partial p_r} \left[ v_1(E_P - \alpha E_r) \right] + v_2(E_P) = 0
\]

(37)

\[
\frac{\partial J_P}{\partial p_c} + \frac{\partial}{\partial p_c} \left[ v_1(E_P - \alpha E_r) \right] + v_2(E_P) = 0
\]

(38)

\[
\frac{\partial J_P}{\partial T_r} + \frac{\partial}{\partial T_r} \left[ v_1(E_P - \alpha E_r) \right] + v_2(E_P) = 0
\]

(39)

\[v_1(E_P - \alpha E_r) = 0 \]

(40)

\[v_2(E_P) = 0 \]

(41)

\[E_P - \alpha E_r = 0 \]

(42)

$E_P \geq 0$

where $v_1$ and $v_2$ are Lagrange multipliers.

By using mathematical software to solve the KKT conditions, we can find optimal $p^*$, $p^*$, $T^*_r$ values for specified values of $I$ and $\alpha$. With known $p^*$, $p^*$ and $T^*_r$, we can derive the optimal joint profit $JP(p^*, p^*, I, T^*_r)$.

(c) Derive the optimal value of $I$ denoted by $\bar{I}$ that satisfies

\[JP(p^*, p^*, I, T^*_r) \leq JP(p^*, p^*, I', T^*_r) \leq JP(p^*, p^*, I' + 1, T^*_r) \]

(44)

With known $I'$, the optimal value $n^*$ and $m^*$ can be obtained by (23) and (24)

Hence the optimal value $p^*$, $p^*$, $n^*$, $m^*$, and $T^*_r$ are derived. The results are illustrated in the following numerical examples.

5. Numerical Example

Example 1 To illustrate the result of the above theory, the parameters used to illustrate the concept are as follows:

The customer annual demand rate, $d = ae^{-bt}$ units per year, where $a = 500$, $b = 0.02$ and the annual return rate, $c = 0.6d$ unit per year; the retailer’s fixed costs to place an order, $A_r = $30; the manufacturer’s manufacturing set up cost, $A_m = $250; the fixed cost to process buyer’s order of any size for manufacturer, $M = $150; the retailer’s and manufacturer’s inventory holding cost rate, $F_r = 0.30$ per unit price per year and $F_m = 0.20$ per unit price per year; the retailer’s and the consumer’s purchase unit price before quantity discounted, $P_r = $45 and $P_c = $65; the manufacturer purchase unit price from the supplier, $P_m = $20; the manufacturer purchase unit price from the consumer, $P_n = 0.25P_r$; the annual production and reproduction rates of the manufacturer, $P = 800$ unit per year; $R = 600$ unit per year; the deterioration rate, $\theta = 0.1$; the negotiation factor, $\alpha = 1$.

What are the values of $p^*$, $p^*$, $T^*_r$, $m^*, n^*$, $JP(p^*, p^*, m^*, n^*, T^*_r)$? And what is the percentage of the joint extra profit ($JP_E$) in this model?

Substitute the above parameters into (27), we can obtain the $T^*_r = 0.1556$ year, and satisfies (30), can derive the $I'$ is 10 and the values $n^* = 5.38$ and $m^* = 4.62$ can be obtained by (23) and (24). Substitute the above parameters into (36). The optimization problem is stated as Appendix.

For the given $I = 6$, from the constraints of the problem ($E_P = 0$, and $E_P \geq 0$), the KKT condition can be derived and the KKT point is $(p^*_r, p^*_c, T^*_r, v_1, v_2) = (41.06, 62.93, 0.372, -0.0270, 0)$, which can be obtained. Substitute the value of $I$, $p^*_r$, $p^*_c$ and $T^*_r$ into (47), (46) and (45), the value of $n^*$ and $m^*$ into the numbers of $I'$ are given in Table 1.
Table 1: The Results of I Value for A Range of Vicinity of The Numbers of $I^*$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$n$</th>
<th>$m$</th>
<th>$T_r^*$</th>
<th>$P_r^*$</th>
<th>$P_e^*$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$TP_r$</th>
<th>$TP_e$</th>
<th>$JP^*$</th>
<th>PEJP</th>
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</thead>
<tbody>
<tr>
<td>10</td>
<td>4.92</td>
<td>5.08</td>
<td>0.293</td>
<td>$42.51$</td>
<td>$63.64$</td>
<td>-0.0214</td>
<td>0</td>
<td>2504.2</td>
<td>2462.2</td>
<td>4966.4</td>
<td>7.05%</td>
</tr>
<tr>
<td>8</td>
<td>4.02</td>
<td>3.98</td>
<td>0.325</td>
<td>$41.84$</td>
<td>$63.30$</td>
<td>-0.0237</td>
<td>0</td>
<td>2545.4</td>
<td>2503.5</td>
<td>5048.9</td>
<td>8.82%</td>
</tr>
<tr>
<td>7</td>
<td>3.56</td>
<td>3.44</td>
<td>0.347</td>
<td>$41.46$</td>
<td>$63.11$</td>
<td>-0.0252</td>
<td>0</td>
<td>2565.8</td>
<td>2523.8</td>
<td>5089.6</td>
<td>9.70%</td>
</tr>
<tr>
<td>6</td>
<td>3.09</td>
<td>2.91</td>
<td>0.372</td>
<td>$41.06$</td>
<td>$62.93$</td>
<td>-0.0270</td>
<td>0</td>
<td>2585.5</td>
<td>2543.6</td>
<td>5129.0</td>
<td>10.55%</td>
</tr>
<tr>
<td>5</td>
<td>2.61</td>
<td>2.39</td>
<td>0.405</td>
<td>$40.62$</td>
<td>$62.73$</td>
<td>-0.0293</td>
<td>0</td>
<td>2603.7</td>
<td>2561.8</td>
<td>5165.5</td>
<td>11.34%</td>
</tr>
<tr>
<td>4</td>
<td>2.12</td>
<td>1.88</td>
<td>0.449</td>
<td>$40.12$</td>
<td>$62.54$</td>
<td>-0.0323</td>
<td>0</td>
<td>2618.8</td>
<td>2576.9</td>
<td>5195.8</td>
<td>11.99%</td>
</tr>
</tbody>
</table>

The optimal value $I^*$ is 3, which satisfies

$JP(p_1^*, p_2^*, T_r^*) = JP(p_1^*, p_3^*, T_r^*) = JP(p_1^*, p_4^*, T_r^*)$ (52)

The optimal retailer’s selling price, $p_1^*$, is $31.6940$, the optimal retailer’s purchase unit price, $p_2^*$, is $21.0408$, the number of deliveries per reproduction cycle time from the manufacturer to the retailer, $n$ is 1.62, the number of deliveries per production cycle time from the manufacturer to the retailer, $m^*$ is 1.38 and the optimal retailer’s ordering interval, $T_r^*$, is 0.513 years. With a known optimal values of $p_1^*$, $p_2^*$, $n$, $m^*$ and $T_r^*$, the following can be obtained:

1. the maximal joint profit $JP(p_1^*, p_2^*, n^*, m^*, T_r^*)$ is $5,212.1$;
2. the extra profits, $EP_r = EP_n = 286.3$;
3. the percentage of the joint extra profit ($JP$,) is 12.34%;
4. comparing the results before pricing discount with the result of pricing discount are given in Table 2.

Table 2: The Results of Comparing The Pricing Discount with No Pricing Discount in Example 1

<table>
<thead>
<tr>
<th>Relative variable of model</th>
<th>$I$</th>
<th>$n$</th>
<th>$m$</th>
<th>$T_r$</th>
<th>$p_r$</th>
<th>$p_e$</th>
<th>$TP_r$</th>
<th>$TP_e$</th>
<th>$JP$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before pricing discount</td>
<td>10</td>
<td>5.38</td>
<td>4.62</td>
<td>0.1556</td>
<td>45</td>
<td>65</td>
<td>2340.7</td>
<td>2298.8</td>
<td>4639.5</td>
</tr>
<tr>
<td>Pricing discount</td>
<td>3</td>
<td>1.62</td>
<td>1.38</td>
<td>0.5126</td>
<td>39.5</td>
<td>62.4</td>
<td>2627.0</td>
<td>2585.1</td>
<td>5212.1</td>
</tr>
</tbody>
</table>

Percentage pricing discount and extra profit -12.22% -4.00% 12.23% 12.45% 12.34%

Example 2 Similar to the parameters given in Example 1, we try to find the values of $n^*$, $m^*$, $p_r^*$, $p_e^*$, $T_r^*$, $JP^*$ and $PJCR$ in the model when the deteriorating rates are 0.001, 0.05, 0.1, 0.15, 0.2 and 0.3.

By the solution procedure, the results are shown in Table 3.

Table 3: The Results for Various Deteriorating Rate Solution

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$n^*$</th>
<th>$m^*$</th>
<th>$T_r^*$</th>
<th>$p_r^*$</th>
<th>$p_e^*$</th>
<th>$JP^*$</th>
<th>PEJP</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>2.40</td>
<td>1.60</td>
<td>0.5048</td>
<td>39.82</td>
<td>60.90</td>
<td>5336.6</td>
<td>8.91%</td>
</tr>
<tr>
<td>0.05</td>
<td>1.70</td>
<td>1.30</td>
<td>0.5373</td>
<td>39.58</td>
<td>61.70</td>
<td>5269.3</td>
<td>10.57%</td>
</tr>
<tr>
<td>0.1</td>
<td>1.62</td>
<td>1.38</td>
<td>0.5127</td>
<td>39.54</td>
<td>62.36</td>
<td>5212.1</td>
<td>12.34%</td>
</tr>
<tr>
<td>0.15</td>
<td>1.55</td>
<td>1.45</td>
<td>0.4904</td>
<td>39.48</td>
<td>62.93</td>
<td>5157.3</td>
<td>14.08%</td>
</tr>
<tr>
<td>0.2</td>
<td>1.02</td>
<td>0.98</td>
<td>0.5809</td>
<td>38.31</td>
<td>63.16</td>
<td>5109.9</td>
<td>15.87%</td>
</tr>
<tr>
<td>0.3</td>
<td>0.96</td>
<td>1.04</td>
<td>0.5461</td>
<td>37.85</td>
<td>63.96</td>
<td>5030.5</td>
<td>19.70%</td>
</tr>
</tbody>
</table>

As shown in Table 4, when the deterioration rate ($\theta$) increases, the optimal retailer’s selling price and the percentage of the extra joint profit (PEJP) also increase, but the optimal retailer’s purchase unit price and the optimal joint profit decrease.

Example 3 From the information given in Example 1, what solutions are various $\alpha$ values?

The results of various $\alpha$ value solutions are given in Table 4. As shown in Table 4, when $\alpha = 0.001$, the retailer and manufacturer’s joint profit is maximized. However, the extra profit is accrued mainly to the retailer. The optimal retailer’s selling price is $62.42$, the optimal retailer’s purchase unit price is $37.57$ and the optimal retailer’s ordering cycle time is 0.5219 years, the maximal joint profit is $5234.1$ and the PEJP value is 12.82%.

Table 4: The Results of Various $\alpha$ Values

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n^*$</th>
<th>$m^*$</th>
<th>$T_r^*$</th>
<th>$p_r^*$</th>
<th>$p_e^*$</th>
<th>$JP^*$</th>
<th>PEJP</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>2.40</td>
<td>1.60</td>
<td>0.5048</td>
<td>39.82</td>
<td>60.90</td>
<td>5336.6</td>
<td>8.91%</td>
</tr>
<tr>
<td>0.05</td>
<td>1.70</td>
<td>1.30</td>
<td>0.5373</td>
<td>39.58</td>
<td>61.70</td>
<td>5269.3</td>
<td>10.57%</td>
</tr>
<tr>
<td>0.1</td>
<td>1.62</td>
<td>1.38</td>
<td>0.5127</td>
<td>39.54</td>
<td>62.36</td>
<td>5212.1</td>
<td>12.34%</td>
</tr>
<tr>
<td>0.15</td>
<td>1.55</td>
<td>1.45</td>
<td>0.4904</td>
<td>39.48</td>
<td>62.93</td>
<td>5157.3</td>
<td>14.08%</td>
</tr>
<tr>
<td>0.2</td>
<td>1.02</td>
<td>0.98</td>
<td>0.5809</td>
<td>38.31</td>
<td>63.16</td>
<td>5109.9</td>
<td>15.87%</td>
</tr>
<tr>
<td>0.3</td>
<td>0.96</td>
<td>1.04</td>
<td>0.5461</td>
<td>37.85</td>
<td>63.96</td>
<td>5030.5</td>
<td>19.70%</td>
</tr>
</tbody>
</table>

As shown in Table 4, when the deterioration rate ($\theta$) increases, the optimal retailer’s selling price and the percentage of the extra joint profit (PEJP) also increase, but the optimal retailer’s purchase unit price and the optimal joint profit decrease.
6. Conclusion

This study considers an optimal replenishment and pricing policy for a closed-loop supply chain inventory system. An inventory supply chain model coordinated through price discounting is developed when the demand is exponentially decreasing. The right price discount and the optimal joint profit are achieved through maximizing the joint profit of the two-echelon inventory system under a best ordering policy are the strategies that can be incorporated into the supply chain management system.

Acknowledgements

This paper was supported in part by the National Science Council, Taiwan, R.O.C., under NSC 95-2221-E-129 -007.

References


Appendix

The optimization problem of Example 1 is stated as

$$\text{max} \quad J_P(p_r, p_c, n, m, T_r) = 500 P e^{-0.02 P_r} \left( \frac{30}{T_r} - \frac{1500 P_r \left( 10 e^{-0.02 P_r} e^{0.1 T_r} - 10 e^{-0.02 P_r} e^{-0.02 P_r} T_r \right)}{T_r} \right)$$

$$- 400 + 150(n + m) - \frac{(0.05n P_c + 4m)}{(n + m)^2} \left( 25000 \left( e^{0.1 T_r} - 1 \right) e^{-0.04 P_c} \right)$$

$$+ \frac{80000 \ln \left( 1 - \frac{5}{6} e^{-0.02 P_r} + \frac{5}{6} e^{-0.02 P_r} e^{0.1 T_r} \right)}{T_r} + \frac{80000 \ln \left( 1 - \frac{5}{6} e^{-0.02 P_c} + \frac{5}{6} e^{-0.02 P_c} e^{0.1 T_r} \right)}{T_r}$$

$$- \frac{50000(n + m) e^{-0.02 P_r} \left( e^{0.1 T_r} - 1 \right) + 25000 \left( e^{0.1 T_r} - 1 \right) e^{-0.04 P_c}}{8 + 5e^{-0.02 P_r} - 5e^{-0.02 P_r} \left( e^{0.1 T_r} \right)}$$

$$- \frac{10000}{(n + m) T_r} \left( 0.15 \ln \left( 1 - \frac{5}{6} e^{-0.02 P_r} + \frac{5}{6} e^{-0.02 P_r} e^{0.1 T_r} \right) + 16 \ln \left( 1 - \frac{5}{6} e^{-0.02 P_c} + \frac{5}{6} e^{-0.02 P_c} e^{0.1 T_r} \right) \right)$$

s. t. \( EP_m = EP_r \) (A2)

\[ EP_r \geq 0 \] (A3)

\[ n = \frac{0.2 \left( 50 \ln \left( 1.2 e^{0.9 e^{-0.02 P_r}} \left( e^{-0.02 P_c} \right) \right) - 1.2 + e^{-0.02 P_r} \right)}{T_r} \] (A4)

\[ m = I - n \] (A5)

\[ I = 1, 2, 3, \ldots \]

\[ p_r, p_c, T_r \geq 0 \]

where

\[ EP_r = 500 P e^{-0.02 P_r} - \frac{30}{T_r} - \frac{1500 P_r \left( 10 e^{-0.02 P_r} e^{0.1 T_r} - 10 e^{-0.02 P_r} e^{-0.02 P_r} T_r \right)}{T_r} - \frac{5000 P e^{-0.02 P_r} \left( e^{0.1 T_r} - 1 \right)}{T_r} - 234070 \] (A6)

\[ EP_m = \frac{5000 P e^{-0.02 P_r} \left( e^{0.1 T_r} - 1 \right) - 400 + 150(n + m)}{(n + m) T_r} - \frac{(0.05n P_c + 4m)}{(n + m)^2} \left( 25000 \left( e^{0.1 T_r} - 1 \right) e^{-0.04 P_c} \right)$$

\[ + \frac{80000 \ln \left( 1 - \frac{5}{6} e^{-0.02 P_r} + \frac{5}{6} e^{-0.02 P_r} e^{0.1 T_r} \right)}{T_r} + \frac{80000 \ln \left( 1 - \frac{5}{6} e^{-0.02 P_c} + \frac{5}{6} e^{-0.02 P_c} e^{0.1 T_r} \right)}{T_r} \]

\[ - \frac{50000(n + m) e^{-0.02 P_r} \left( e^{0.1 T_r} - 1 \right) + 25000 \left( e^{0.1 T_r} - 1 \right) e^{-0.04 P_c}}{8 + 5e^{-0.02 P_r} - 5e^{-0.02 P_r} \left( e^{0.1 T_r} \right)} $$

\[ - \frac{10000}{(n + m) T_r} \left( 0.15 \ln \left( 1 - \frac{5}{6} e^{-0.02 P_r} + \frac{5}{6} e^{-0.02 P_r} e^{0.1 T_r} \right) + 16 \ln \left( 1 - \frac{5}{6} e^{-0.02 P_c} + \frac{5}{6} e^{-0.02 P_c} e^{0.1 T_r} \right) \right) \] (A7)